

GLOBAL WELL-POSEDNESS FOR THE k -DISPERSION GENERALIZED BENJAMIN-ONO EQUATION

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ABSTRACT. We consider the k -dispersion generalized Benjamin-Ono equation in the supercritical case. We establish sharp conditions on the data to show global well-posedness in the energy space for this family of nonlinear dispersive equations. We also prove similar results for the generalized Benjamin-Ono equation.

1. INTRODUCTION

In this paper we continue the study on sharp conditions to obtain global solutions for supercritical nonlinear dispersive models. With this aim we consider the initial value problem (IVP) for the k -dispersion generalized Benjamin-Ono (DGBO) equation

$$\begin{cases} \partial_t u - D^\beta \partial_x u + \partial_x(u^{k+1}) = 0, & x \in \mathbb{R}, t > 0, \quad 1 \leq \beta < 2, \quad k \in \mathbb{Z}^+, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.1)$$

where D^β denotes the homogeneous derivative of order $\beta \in \mathbb{R}$,

$$D^\beta f = c_\beta(|\xi|^\beta \widehat{f})^\vee = (\mathcal{H} \partial_x)^\beta f,$$

and \mathcal{H} denotes the Hilbert transform,

$$\mathcal{H}f(x) = (-i \operatorname{sgn}(\xi) \widehat{f}(\xi))^\vee(x).$$

These equations arise as mathematical models for the unidirectional propagation of weakly nonlinear dispersive long waves. Notice that when $\beta = 2$ and $k = 1$ the equation in (1.1) is the well known Korteweg-de Vries (KdV) equation and when $\beta = 1$ and $k = 1$ it is the Benjamin-Ono (BO) equation. (See [26], [3] and [34] for their derivation and [23], [5], [17] for the sharpest local and global well-posedness results).

We shall mention that recently several developments regarding the local well-posedness has been obtained for both BO and DGBO equations. One of the interesting features of the IVP associated to these families of equations is the obstruction to its solvability by iteration methods. Indeed, Molinet, Saut and Tzvetkov in [33] proved that the contraction mapping principle cannot be applied to solve these equations in the standard Sobolev spaces. For more details concerning well-posedness we refer to [22], [6], [32], [14], [13], [15], [8], [9], [31] for the GDBO equation and [4], [11], [10], [18], [17], [19], [25], [29], [35], [36], [37] for the BO equation.

Regarding local well-posedness for the IVP (1.1), $k \geq 2$, in [22] Kenig, Ponce and Vega showed that the IVP (1.1) is locally well-posed in $H^s(\mathbb{R})$, for $s \geq (9 - 3\beta)/4$ and globally well-posed in $H^{\beta/2}(\mathbb{R})$ for $\beta \geq 9/5$.

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Recently, Kenig, Martel and Robiano [20] considered the family of dispersion generalized Benjamin-Ono equations,

$$\partial_t u - D^\beta \partial_x u + |u|^{2\beta} \partial_x u = 0, \quad (1.2)$$

which are critical with respect to both the L^2 norm and the global existence of solutions. Observe that these equations are the interpolated ones between the modified BO equation ($\beta = 1$) and the critical KdV equation $\beta = 2$. In [20] it was proved among other results, the local well-posedness in the energy space $H^{\beta/2}(\mathbb{R})$ for $\beta \in (1, 2)$. They also studied the blow-up problem in the same vein as in [27] and [28]. In particular they established the finite time blow-up of solutions with negative energy in the energy space when β is close to 2.

Here, we are interested in global well-posedness for the DGBO equation in the supercritical case, that is, $k > 2\beta$. Important relations in this study are the conserved quantities satisfied by real solutions of the IVP (1.1)

$$M(u(t)) = \int_{\mathbb{R}} u^2(x, t) dx \quad (1.3)$$

and

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}} |D^{\beta/2} u(x, t)|^2 dx - \frac{1}{k+2} \int_{\mathbb{R}} u^{k+2}(x, t) dx. \quad (1.4)$$

These quantities allow us to establish an *a priori* estimate in the energy space $H^{\beta/2}(\mathbb{R})$.

Next we will describe our main results concerning the case $\beta \in [1, 2)$. First, we recall that in [7] the authors studied the IVP associated with the L^2 -supercritical generalized KdV equation ($\beta = 2$ in equation (1.1)). Under sharp conditions satisfied by the initial data they established global well-posedness in the energy space $H^1(\mathbb{R})$. This result is the first step in view to the more general theory studied by Kenig and Merle [21] and by Holmer and Roudenko [16]. Our study below is based on these works.

One of the main tools needed for this analysis is having the best constant for the Gagliardo-Nirenberg inequality

$$\|f\|_{L^{k+2}}^{k+2} \leq K_{\text{opt}}^{k+2} \|D^{\beta/2} f\|_{L^2}^{k/\beta} \|f\|_{L^2}^{(2+(k+2)(\beta-1))/\beta},$$

where the constant K_{opt} depends on Q the unique positive, even, decreasing (for $x > 0$) solution of the equation

$$D^\beta Q + Q - Q^{k+1} = 0. \quad (1.5)$$

The sharp constant was obtained in [1]. In [39] the existence of the ground state Q was showed. The uniqueness of Q which is also key in our arguments was recently established by Frank and Lenzmann in [12].

Our main result reads as follows.

Theorem 1.1. *Let $\beta \in [1, 2)$ and $u_0 \in H^{\beta/2}(\mathbb{R})$. Assume $k > 2\beta$ and let $s_k = 1/2 - \beta/k$. Suppose that*

$$E(u_0)^{s_k} M(u_0)^{\beta/2-s_k} < E(Q)^{s_k} M(Q)^{\beta/2-s_k}, \quad E(u_0) \geq 0. \quad (1.6)$$

If

$$\|D^{\beta/2} u_0\|_{L^2}^{s_k} \|u_0\|_{L^2}^{\beta/2-s_k} < \|D^{\beta/2} Q\|_{L^2}^{s_k} \|Q\|_{L^2}^{\beta/2-s_k}, \quad (1.7)$$

then for any t as long as the solution exists,

$$\|D^{\beta/2}u(t)\|_{L^2}^{s_k}\|u_0\|_{L^2}^{\beta/2-s_k} = \|D^{\beta/2}u(t)\|_{L^2}^{s_k}\|u(t)\|_{L^2}^{\beta/2-s_k} < \|D^{1/2}Q\|_{L^2}^{s_k}\|Q\|_{L^2}^{\beta/2-s_k}, \quad (1.8)$$

where Q is the unique positive, even, decreasing (for $x > 0$) solution of (1.5).

Corollary 1.2. *Let $\beta \in (1, 2)$. Under the hypotheses of Theorem 1.1 the local solutions given in Theorem 2.3 below can be extended to any interval of time $[0, T]$.*

Next we specialize ourselves in the case $\beta = 1$, thus the equation in (1.1) becomes the generalized BO equation for $k \geq 2$ and the energy space is $H^{1/2}(\mathbb{R})$. A scaling argument suggests that the best local well-posedness for the IVP (1.1) should be attained for $s > s_k = 1/2 - 1/k$. We review the state of the art concerning this IVP. In the case $k = 2$, Kenig and Takaoka [24] established local well-posedness for data in $H^{1/2}(\mathbb{R})$ and, using the conserved quantities, global well-posedness in $H^s(\mathbb{R})$, $s \geq 1/2$. For $k = 3$ and $k \geq 4$, local-wellposedness was obtained by Vento in [38] in $H^s(\mathbb{R})$, $s > 1/3$ and $s \geq s_k$, respectively. We shall observe that in these results the existence time depends on the initial data itself and not only on its norm, that is, $T = T(u_0)$. Therefore, we cannot use the conserved quantities to extend the local solutions to global ones by means of *a priori* estimates. However, if $k \geq 5$, the local theory was proved in $H^s(\mathbb{R})$, $s \geq 1/2$ with $T = T(\|u_0\|_s) > 0$ in [31]. Using these local solutions we can extend them globally as stated in the next result.

Corollary 1.3. *Assume $k \geq 5$ and the hypotheses of Theorem 1.1 for $\beta = 1$. Then the local $H^{1/2}$ -solution of the generalized Benjamin-Ono (gBO) equation*

$$\begin{cases} u_t - \mathcal{H}\partial_x^2 u + \partial_x(u^{k+1}) = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.9)$$

can be extended to any interval of time $[0, T]$.

Remark 1.4. *We observe that in the cases $k = 3, 4$ it is an open problem to establish a local theory in the energy space with time $T = T(\|u_0\|_{1/2})$. With this result available Theorem 1.1 will imply in the existence of global solutions with the sharp conditions on the data.*

The paper is organized as follows: In Section 2 the local theory for the IVP (1.1), $\beta \in (1, 2)$ and $k > 2\beta$ will be established. Then, in Section 3, the proof of Theorem 1.1 will be given.

2. LOCAL WELL-POSEDNESS IN THE ENERGY SPACE

Here we consider the Cauchy problem associated with the supercritical dispersion generalized Benjamin-Ono (DGBO) equation

$$\begin{cases} u_t - D^\beta \partial_x u + \partial_x(u^{k+1}) = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (2.10)$$

where u is a real-valued function, $1 < \beta < 2$ and $k > 2\beta$ is an integer number.

First, consider the linear IVP

$$\begin{cases} u_t - D^\beta \partial_x u = 0, & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ u(x, 0) = u_0(x). \end{cases} \quad (2.11)$$

The solution of (2.11) is given by the unitary group $\{U_\beta(t)\}_{t=-\infty}^\infty$ such that

$$u(t) = U_\beta(t)u_0(x) = \int_{\mathbb{R}} e^{i(t|\xi|^\beta \xi + x\xi)} \widehat{u}_0(\xi) d\xi. \quad (2.12)$$

We start by recalling the following estimates proved in [20].

Lemma 2.1. *Assume $1 < \beta < 2$. For $0 < T < 1$, there exists a constant $C > 0$ such that*

(i) *For all $u_0 \in L^2$,*

$$\|U_\beta(t)u_0\|_{L_T^\infty L_x^2} \leq C\|u_0\|_{L^2}.$$

(ii) *There exists $\gamma > 0$ such that for all $u_0 \in H^{\beta/2}$,*

$$\|\partial_x U_\beta(t)u_0\|_{L_x^\infty L_T^2} \leq CT^\gamma \|u_0\|_{H^{\beta/2}}.$$

(iii) *For all $u_0 \in H^s$, where $s > \frac{3}{4} - \frac{\beta}{4}$,*

$$\|U_\beta(t)u_0\|_{L_x^{2\beta} L_T^\infty} \leq C\|u_0\|_{H^{s+}}.$$

(iv) *For all $g \in L_x^1 L_T^2$,*

$$\|D^{\beta/2} \int_0^t U_\beta(t-t')g(\cdot, t')dt'\|_{L_T^\infty L_x^2} \leq C\|g\|_{L_x^1 L_T^2}.$$

(v) *For $0 \leq s < \beta$, there exists $\gamma > 0$ such that*

$$\|D^s \int_0^t U_\beta(t-t')g(\cdot, t')dt'\|_{L_x^\infty L_T^2} \leq CT^\gamma \|g\|_{L_x^1 L_T^2}.$$

(vi) *There exists $\gamma > 0$ such that*

$$\left\| \int_0^t U_\beta(t-t')g(\cdot, t')dt' \right\|_{L_T^\infty L_x^2} \leq CT^\gamma \|g\|_{L_x^1 L_T^2}.$$

Proof. See Lemma 1 in [20]. □

Corollary 2.2. *Assume $1 < \beta < 2$, $k > 2\beta$ and let $1/2+$ denotes any number bigger than $1/2$. The following statements hold.*

(i) *For all $u_0 \in H^{1/2+}$,*

$$\|U_\beta(t)u_0\|_{L_x^k L_T^\infty} \leq C\|u_0\|_{H^{1/2+}}$$

(ii) *For all $g \in L_x^1 L_T^2$ such that $\partial_x g \in L_x^1 L_T^2$,*

$$\left\| \int_0^t U_\beta(t-t')\partial_x g(\cdot, t')dt' \right\|_{L_x^k L_T^\infty} \leq C\|\partial_x g\|_{L_x^1 L_T^2}.$$

Proof. From Lemma 2.1-(iii), we have

$$\|U_\beta(t)u_0\|_{L_x^{2\beta} L_T^\infty} \leq C\|u_0\|_{H^{1/2+}}. \quad (2.13)$$

Also, from Sobolev's embedding,

$$\|U_\beta(t)u_0\|_{L_x^\infty L_T^\infty} \leq C\|u_0\|_{H^{1/2+}}. \quad (2.14)$$

Hence, (i) follows just interpolating (2.13) and (2.14).

The proof of (ii) follows exactly as in [20, Lemma 1-(xi)]. Indeed, let P_n be the projection on frequencies $\simeq 2^n$ and define

$$T_n g(\cdot, t) = \int_0^t U_\beta(t-t')P_n \partial_x g(\cdot, t')dt',$$

$$\tilde{T}_n g(\cdot, t) = \int_0^T U_\beta(t-t') P_n \partial_x g(\cdot, t') dt'.$$

Choose a number $1/2+$ such that $1/2+ < \beta/2$. Then, by (i), localization in frequencies, and Lemma 2.1-(iv), we deduce

$$\begin{aligned} \|\tilde{T}_n g\|_{L_x^k L_T^\infty} &= \|U_\beta(t) \int_0^T U_\beta(-t') P_n \partial_x g(\cdot, t') dt'\|_{L_x^k L_T^\infty} \\ &\leq C 2^{n((1/2+)-\beta/2)} \|D^{\beta/2} \int_0^T U_\beta(-t') P_n \partial_x g(\cdot, t') dt'\|_{L_x^2} \\ &\leq C 2^{n((1/2+)-\beta/2)} \|D^{\beta/2} \int_0^T U_\beta(T-t') P_n \partial_x g(\cdot, t') dt'\|_{L_x^2} \\ &\leq C 2^{n((1/2+)-\beta/2)} \|\partial_x g\|_{L_x^1 L_T^2}. \end{aligned}$$

The conclusion then follows from the Christ and Kiselev lemma (see [30, Lemma 3]), as in [20]. \square

Now we are able to prove the following well-posedness result.

Theorem 2.3. *Let $1 < \beta < 2$ and $k > 2\beta$. For any $u_0 \in H^{\beta/2}(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^{\beta/2}}) > 0$ and a unique solution of the IVP (2.10), defined in the interval $[0, T]$, such that*

$$u \in C([0, T]; H^{\beta/2}(\mathbb{R})), \quad (2.15)$$

$$\|\partial_x u\|_{L_x^\infty L_T^2} < \infty, \quad (2.16)$$

and

$$\|u\|_{L_x^k L_T^\infty} < \infty. \quad (2.17)$$

Moreover, for any $T' \in (0, T)$ there exists a neighborhood U of u_0 in $H^{\beta/2}(\mathbb{R})$ such that the map $\tilde{u}_0 \mapsto \tilde{u}(t)$ from U into the class defined by (2.15)–(2.17) is continuous.

Proof. As usual, we consider the integral operator

$$\Psi(u)(t) = \Psi_{u_0}(u)(t) := U_\beta(t)u_0 - \int_0^t U_\beta(t-t') \partial_x(u^{k+1})(t') dt', \quad (2.18)$$

and define the metric spaces

$$\mathcal{Y}_T = \{u \in C([0, T]; H^{\beta/2}(\mathbb{R})); \ \|u\| < \infty\}$$

and

$$\mathcal{Y}_T^a = \{u \in \mathcal{Y}_T; \ \|u\| \leq a\},$$

with

$$\|u\| := \|u\|_{L_T^\infty H_x^{\beta/2}} + \|u\|_{L_x^k L_T^\infty} + T^{-\gamma} \|u_x\|_{L_x^\infty L_T^2}$$

where $\gamma > 0$ is an arbitrarily small number and $a, T > 0$ will be chosen later. We assume $T \leq 1$.

First we estimate the $H^{\beta/2}$ -norm of $\Psi(u)$. Let $u \in \mathcal{Y}_T$. By using Lemma 2.1 (i) and (vi), and then Hölder's inequality, we have

$$\begin{aligned} \|\Psi(u)(t)\|_{L_T^\infty L_x^2} &\leq \|U_\beta(t)u_0\|_{L_T^\infty L_x^2} + \left\| \int_0^t U_\beta(t-t')\partial_x(u^{k+1})(t')dt' \right\|_{L_T^\infty L_x^2} \\ &\leq C\|u_0\|_{L_x^2} + C\|u^k u_x\|_{L_x^1 L_T^2} \\ &\leq C\|u_0\|_{L_x^2} + CT^\gamma \|u\|_{L_x^k L_T^\infty}^k T^{-\gamma} \|u_x\|_{L_x^\infty L_T^2} \\ &\leq C\|u_0\|_{L_x^2} + CT^\gamma \|u\|^{k+1}. \end{aligned} \quad (2.19)$$

Also, using Lemma 2.1 (i) and (iv), and Hölder's inequality, we get

$$\begin{aligned} \|D^{\beta/2}\Psi(u)(t)\|_{L_T^\infty L_x^2} &\leq \|D^{\beta/2}U_\beta(t)u_0\|_{L_T^\infty L_x^2} + \|D^{\beta/2} \int_0^t U_\beta(t-t')\partial_x(u^{k+1})(t')dt' \|_{L_T^\infty L_x^2} \\ &\leq C\|D^{\beta/2}u_0\|_{L_x^2} + C\|u^k u_x\|_{L_x^1 L_T^2} \\ &\leq C\|u_0\|_{H^{\beta/2}} + CT^\gamma \|u\|^{k+1}. \end{aligned} \quad (2.20)$$

Now, in view of Lemma 2.1 (ii) and (v) (with $s = 1$),

$$\begin{aligned} \|\partial_x \Psi(u)(t)\|_{L_x^\infty L_T^2} &\leq \|\partial_x U_\beta(t)u_0\|_{L_x^\infty L_T^2} + \left\| \partial_x \int_0^t U_\beta(t-t')\partial_x(u^{k+1})(t')dt' \right\|_{L_x^\infty L_T^2} \\ &\leq C\|u_0\|_{H^{\beta/2}} + C\|u^k u_x\|_{L_x^1 L_T^2} \\ &\leq C\|u_0\|_{H^{\beta/2}} + CT^\gamma \|u\|^{k+1}. \end{aligned} \quad (2.21)$$

Finally, taking a number $1/2+ < \beta/2$, an application of Corollary 2.2 yields

$$\begin{aligned} \|\Psi(u)(t)\|_{L_x^k L_T^\infty} &\leq \|U_\beta(t)u_0\|_{L_x^k L_T^\infty} + \left\| \int_0^t U_\beta(t-t')\partial_x(u^{k+1})(t')dt' \right\|_{L_x^k L_T^\infty} \\ &\leq C\|u_0\|_{H^{1/2+}} + C\|u^k u_x\|_{L_x^1 L_T^2} \\ &\leq C\|u_0\|_{H^{\beta/2}} + CT^\gamma \|u\|^{k+1}. \end{aligned} \quad (2.22)$$

Choose $a = 6C\|u_0\|_{H^{\beta/2}}$, and $T > 0$ such that

$$C a^k T^\gamma \leq \frac{1}{6}.$$

Then, we see that $\Psi : \mathcal{Y}_T^a \mapsto \mathcal{Y}_T^a$ is well defined. Moreover, similar arguments show that Ψ is a contraction. The rest of the proof uses standard arguments, thus, we omit the details. \square

3. GLOBAL WELL-POSEDNESS IN THE ENERGY SPACE

Let us start by recalling the following sharp Gagliardo-Nirenberg type inequality.

Theorem 3.1. *Let $k > 0$ and $1 < \beta < 2$, then the Gagliardo-Nirenberg inequality*

$$\|f\|_{L^{k+2}}^{k+2} \leq K_{\text{opt}}^{k+2} \|D^{\beta/2} f\|_{L^2}^{k/\beta} \|f\|_{L^2}^{(2+(k+2)(\beta-1))/\beta}, \quad (3.23)$$

holds, and the sharp constant $K_{\text{opt}} > 0$ is

$$K_{\text{opt}}^{k+2} = \frac{(k+2)\beta}{2+(k+2)(\beta-1)} \left[\left(\frac{2+(k+2)(\beta-1)}{k} \right)^{1/\beta} \frac{1}{\|Q\|_{L^2}^2} \right]^{k/2}, \quad (3.24)$$

where Q is the unique non-negative, radially-symmetric, decreasing solution of the equation

$$D^\beta Q + Q - Q^{k+1} = 0. \quad (3.25)$$

Proof. The sharpness of the constant in (3.24) was proved in [1, Theorem 2.1]. The uniqueness of the non-negative, radially-symmetric, decreasing (for $x > 0$) solution of (3.25) was recently proved in [12] for the case $\beta \in (1, 2)$. For $\beta = 1$ it was proved by Amick and Toland [2]. \square

Let us establish some useful identities involving the ground state Q . First, by multiplying (3.25) by Q , integrating over \mathbb{R} , we obtain

$$\int_{\mathbb{R}} Q^{k+2} dx = \|Q\|_{L^2}^2 + \|D^{\beta/2} Q\|_{L^2}^2. \quad (3.26)$$

Second, by multiplying (3.25) by $x\partial_x Q$, integrating, and applying integration by parts, we obtain

$$\frac{2}{k+2} \int_{\mathbb{R}} Q^{k+2} dx = \|Q\|_{L^2}^2 - (\beta-1) \|D^{\beta/2} Q\|_{L^2}^2, \quad (3.27)$$

where we have used the identity

$$\int_{\mathbb{R}} x\partial_x Q(D^\beta Q) dx = \frac{\beta-1}{2} \int_{\mathbb{R}} |D^{\beta/2} Q|^2 dx.$$

Combining (3.26) with (3.27), we conclude that

$$\frac{k}{2 + (k+2)(\beta-1)} \|Q\|_{L^2}^2 = \|D^{\beta/2} Q\|_{L^2}^2 \quad (3.28)$$

and

$$\frac{2 + (k+2)(\beta-1)}{k+2} \int_{\mathbb{R}} Q^{k+2} dx = \beta \|Q\|_{L^2}^2. \quad (3.29)$$

With these tools in hand, we are able to prove the following a priori estimate.

Proof of Theorem 1.1. We write the $\dot{H}^{\beta/2}$ -norm of $u(t)$ using the quantities $M(u(t))$ and $E(u(t))$ and then use the sharp Gagliardo-Nirenberg inequality (3.23) to get

$$\begin{aligned} \|D^{\beta/2} u(t)\|_{L^2}^2 &= 2E(u_0) + \frac{2}{k+2} \int_{\mathbb{R}} u^{k+2}(t) dx \\ &\leq 2E(u_0) + \frac{2}{k+2} K_{\text{opt}}^{k+2} \|u_0\|_{L^2}^{(2+(k+2)(\beta-1))/\beta} \|D^{\beta/2} u(t)\|_{L^2}^{k/\beta} \end{aligned} \quad (3.30)$$

Let $X(t) = \|D^{\beta/2} u(t)\|_{L^2}^2$, $A = 2E(u_0)$, and

$$B = \frac{2}{k+2} K_{\text{opt}}^{k+2} \|u_0\|_{L^2}^{(2+(k+2)(\beta-1))/\beta}.$$

Therefore, we can write (3.30) as

$$X(t) - B X(t)^{k/2\beta} \leq A, \text{ for } t \in (0, T). \quad (3.31)$$

Now let $f(x) = x - B x^{k/2\beta}$, for $x \geq 0$. The function f has a local maximum at $x_0 = \left(\frac{2\beta}{kB}\right)^{2\beta/(k-2\beta)}$ with maximum value $f(x_0) = \frac{k-2\beta}{k} \left(\frac{2\beta}{kB}\right)^{2\beta/(k-2\beta)}$. If we require that

$$2E(u_0) < f(x_0) \text{ and } X(0) < x_0, \quad (3.32)$$

the continuity of $X(t)$ implies that $X(t) < x_0$ for any t as long as the solution exists.

Using (3.28) and (3.29), we immediately deduce that

$$E(Q) = \frac{1}{2} \frac{k - 2\beta}{2 + (k + 2)(\beta - 1)} \|Q\|_{L^2}^2. \quad (3.33)$$

Therefore, a simple calculation shows that conditions (3.32) are exactly the inequalities (1.6) and (1.7). Indeed,

$$2E(u_0) < f(x_0) \quad \Leftrightarrow \quad 2E(u_0) < \frac{k - 2\beta}{k} \left(\frac{2\beta}{kB} \right)^{2\beta/(k-2\beta)}$$

Using the explicit form of K_{opt}^{k+2} in (3.24), we have

$$\left(\frac{2\beta}{kB} \right)^{2\beta/(k-2\beta)} = \frac{k}{2 + (k + 2)(\beta - 1)} \frac{\|Q\|_{L^2}^{2k\beta/(k-2\beta)}}{\|u_0\|_{L^2}^{2(2+(k+2)(\beta-1))/(k-2\beta)}}.$$

Thus,

$$\begin{aligned} 2E(u_0) &< f(x_0) \\ \Leftrightarrow \\ E(u_0) \|u_0\|_{L^2}^{2(2+(k+2)(\beta-1))/(k-2\beta)} &< \frac{(k - 2\beta) \|Q\|_{L^2}^{2+2(2+(k+2)(\beta-1))/(k-2\beta)}}{2(2 + (k + 2)(\beta - 1))} \\ \Leftrightarrow E(u_0) \|u_0\|_{L^2}^{2(2+(k+2)(\beta-1))/(k-2\beta)} &< E(Q) \|Q\|_{L^2}^{2(2+(k+2)(\beta-1))/(k-2\beta)} \\ \Leftrightarrow E(u_0)^{s_k} M(u_0)^{\beta/2-s_k} &< E(Q)^{s_k} M(Q)^{\beta/2-s_k}. \end{aligned}$$

One see that the second inequality in (3.32) is equivalent to (1.7) in a similar way. Moreover the inequality $X(t) < x_0$ reduces to (1.8). The proof of Theorem 1.1 is thus completed. \square

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